Reliability Theory of Stochastic Fracture Processes in Sustained Loading: Part II, Stochastic Loading

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A Markovian model of crack growth was presented in the first part of the paper. The present Part II presents a model for the case of random load, in order to obtain the probability distribution of the resulting crack size. A general model and the particular cases of a uniformly distributed stress intensity factor and mechanical work, are presented. This model separates the factors producing the crack growth in a homogeneous material from those considered in the Weibull model for nonhomogeneous materials.

1. Introduction

Several stochastic approaches to the study of fracture processes are based on the Paris-Erdogan linear crack growth law [1], which often leads to the Weibull distribution of the residual strength [2, 3]. An alternative approach is based on the Markovian model of crack growth processes [4, 5], which, under certain conditions, can also lead to the Weibull distribution of the residual strength [5]. The two approaches combine indistinguishably the factors that produce crack propagation in both homogeneous and non-homogeneous materials. Using the results of Part I in which the microscopic crack propagation study is based on the Markovian model [6], the present analysis separates the crack propagation factors of homogeneous materials for the general case of stochastic sustained loading.

2. General Model

Assuming a given probability density function of the stress intensity factor (pdf_K), the probability density function of the crack size (pdf_E) can be obtained in the following manner:

A) Consider a monotonous function

$$W = f(K)\,, \tag{1}$$

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and its inverse function

$$K = F(W)\,, \tag{2}$$

where W is the mechanical work contributed to the crack propagation by the stress and K is the stress intensity factor.

For a given (pdf_K), the corresponding (pdf_W) is

$$pdf_W(W) = \left| \frac{d}{dW} F(W) \right| pdf_K[F(W)]\,. \tag{3}$$

B) For the bond breaking rate function $\mathbf{k}_b$ and the bond healing rate function $\mathbf{k}_h$ [1]

$$\mathbf{k}_b = \frac{kT}{h} \exp\left\{-\frac{\Delta G_b^{\ddagger} - W}{kT}\right\}, \tag{4}$$

$$\mathbf{k}_h = \frac{kT}{h} \exp\left\{-\frac{\Delta G_h^{\ddagger} + W}{kT}\right\}, \tag{5}$$

(where the meaning of the terms are presented in Part I), the inverse functions are

$$W = F_b(\mathbf{k}_b) = \Delta G_b^{\ddagger} + kT \ln \frac{\mathbf{k}_b h}{kT} \tag{6}$$

and

$$W = F_h(\mathbf{k}_h) = -\left(\Delta G_h^{\ddagger} + kT \ln \frac{\mathbf{k}_h h}{kT}\right). \tag{7}$$

The derivatives of (6) and (7) are

$$\frac{dF_b(\mathbf{k}_b)}{d\mathbf{k}_b} = \frac{kT}{\boldsymbol{k}_b} \tag{8}$$

and

$$\frac{dF_h(\mathbf{k}_h)}{d\mathbf{k}_h} = -\frac{kT}{\mathbf{k}_h}\,. \tag{9}$$

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For the probability distribution function of the work, (pdf_W) in (3), the expressions for ($pdf_{\mathbf{k}b}$) and ($pdf_{\mathbf{k}h}$) are

$$pdf_{\mathbf{k}_b}(\mathbf{k}_b) = \left| \frac{dF_b(\mathbf{k}_b)}{d\mathbf{k}_b} \right| pdf_W[F_b(\mathbf{k}_b)]$$
$$= \frac{kT}{\mathbf{k}_b} pdf_W[F_b(\mathbf{k}_b)] \quad (10)$$
$$= \frac{kT}{\mathbf{k}_b} \left| \frac{d}{dW_b} F(W) \right|_{W=F_b(\mathbf{k}_b)} pdf_K\{F[F_b(\mathbf{k}_b)]\},$$

$$pdf_{\mathbf{k}_h}(\mathbf{k}_h) = \left| \frac{dF_h(\mathbf{k}_h)}{d\mathbf{k}_h} \right| pdf_W[F_h(\mathbf{k}_h)]$$
$$= \frac{kT}{h} pdf_W[F_h(\mathbf{k}_h)] \quad (11)$$
$$= \frac{kT}{\mathbf{k}_h} \left| \frac{d}{dW_h} F(W) \right|_{W=F_h(\mathbf{k}_h)} pdf_K\{F[F_h(\mathbf{k}_h)]\}.$$

C) For the average crack size function [7]

$$E = at(\mathbf{k}_b - \mathbf{k}_h), \quad (12)$$

(where E is the average crack size, that is, the expectation value of the average crack size; a is the atomic distance; t is the time;) we introduce the linear transformation

$$y = \mathbf{k}_b - \mathbf{k}_h = E/at, \quad (13)$$

or

$$\mathbf{k}_b = y + \mathbf{k}_h = E/at + \mathbf{k}_h. \quad (14)$$

Accordingly, the derivative of y is

$$dy/dE = 1/at. \quad (15)$$

Assuming a random stress intensity factor with long-wave variations and s-independent $\mathbf{k}_b$ and $\mathbf{k}_h$, (pdf_y) can be derived using the convolution integral

$$pdf_y(y) = \int_0^\infty pdf_{\mathbf{k}_b}(y + \mathbf{k}_h)\, pdf_{\mathbf{k}_h}(\mathbf{k}_h)\, d\mathbf{k}_h, \quad (16)$$

and (pdf_E) is given as

$$pdf_E(E) = \left| \frac{dy}{dE} \right| pdf_y\left(\frac{E}{at}\right)$$
$$= \frac{1}{at} \int_0^\infty pdf_{\mathbf{k}_b}\left(\frac{E}{at} + \mathbf{k}_h\right) pdf_{\mathbf{k}_h}(\mathbf{k}_h)\, d\mathbf{k}_h. \quad (17)$$

Substituting (10) and (11) into (17), the result is

$$pdf_E(E) = \frac{k^2T^2}{at} \int_0^\infty \left| \frac{dF(W)}{dW} \right|_{W=F_b(E/at+\mathbf{k}_h)} \quad (18)$$
$$\cdot \left| \frac{dF(W)}{dW} \right|_{W=F_h(\mathbf{k}_h)} \frac{1}{(E/at + \mathbf{k}_h)\, \mathbf{k}_h}$$
$$\cdot pdf_K \left\{ F \left[\Delta G_b^{\ddagger} + kT \ln \frac{(E/at + \mathbf{k}_h)\, h}{kT} \right] \right\}$$
$$\cdot pdf_K \left[F \left(-\Delta G_h^{\ddagger} - kT \ln \frac{\mathbf{k}_h h}{kT} \right) \right] d\mathbf{k}_h.$$

Equation (18) is the probability distribution function of the average crack size for a lengthy varying random stress intensity factor with the characteristics of $pdf_K(\cdot)$. Because of the deterministically dependent stress and strength assumption, the present model is different from the known model that assumes statistically independent stress and strength. At any given moment of crack propagation under sustained loading, the actual strength heavily depends on the loading history of the specimen.

3. Particular Cases

In order to clarify (18), two particular cases are analyzed.

(a) Consider the uniformly distributed stress intensity factor over the domain $\{\alpha \div \beta\}$:

$$pdf_K(K) = \begin{cases} 1/(\beta - \alpha) & \text{if} \quad \alpha < K \leq \beta, \\ 0 & \text{elsewhere}, \end{cases}$$

and assume, for (1), that the work and its inverse function are respectively,

$$W = f(K) = \mathbf{K}\, K^2; \quad (20)$$

and

$$K = F(W) = (W/\mathbf{K})^{1/2}. \quad (21)$$

The derivative of (21) is

$$\frac{dF(W)}{dW} = \frac{1}{2} \left(\frac{W}{\mathbf{K}} \right)^{-1/2} \quad (22)$$

and (3) becomes

$$pdf_W(W) = \begin{cases} \dfrac{(\mathbf{K})^{1/2}}{2(W)^{1/2}} \dfrac{1}{\beta - \alpha} & \text{if} \quad \mathbf{K}\,\alpha^2 < W \leq \mathbf{K}\,\beta^2, \\ 0 & \text{elsewhere}. \end{cases} \quad (23)$$

Also, $(pdf_{\mathbf{k}_b})$ and $(pdf_{\mathbf{k}_h})$, (10) and (11), take the form

$$pdf_{\mathbf{k}_b}(\mathbf{k}_b) = \begin{cases} \dfrac{kT}{\mathbf{k}_b} \dfrac{(\mathbf{K})^{1/2}}{2\left(\Delta G_b^{\ddagger} + kT \ln \dfrac{\mathbf{k}_b h}{kT}\right)^{1/2}} \dfrac{1}{\beta - \alpha} \\ \quad \text{if} \quad \dfrac{kT}{h} \exp\left(-\dfrac{\Delta G_b^{\ddagger} - \mathbf{K}\,\alpha^2}{kT}\right) < \mathbf{k}_b \leqq \dfrac{kT}{h} \exp\left(-\dfrac{G_b^{\ddagger} - \mathbf{K}\,\beta^2}{kT}\right), \\ 0 \quad \text{elsewhere}; \end{cases} \tag{24}$$

$$pdf_{\mathbf{k}_h}(\mathbf{k}_h) = \begin{cases} \dfrac{kT}{\mathbf{k}_h} \dfrac{(\mathbf{K})^{1/2}}{2\left(-\Delta G_h^{\ddagger} - kT \ln \dfrac{\mathbf{k}_h h}{kT}\right)^{1/2}} \dfrac{1}{\beta - \alpha} \\ \quad \text{if} \quad \dfrac{kT}{h} \exp\left(-\dfrac{\Delta G_h^{\ddagger} + \mathbf{K}\,\beta^2}{kT}\right) \leqq \mathbf{k}_h < \dfrac{kT}{h} \exp\left(-\dfrac{\Delta G_h^{\ddagger} + \mathbf{K}\,\alpha^2}{kT}\right), \\ 0 \quad \text{elsewhere}. \end{cases} \tag{25}$$

In this case, (18), the probability distribution function of the average crack size (pdf_E), becomes

$$pdf_E(E) = \begin{cases} \dfrac{(\mathbf{K})^{1/2} k^2 T^2}{at(\beta-\alpha)^2} \displaystyle\int_0^{\infty} \dfrac{1}{4\mathbf{k}_h\left(\dfrac{E}{at} + \mathbf{k}_h\right)} \dfrac{d\mathbf{k}_h}{\left\{\left[\Delta G_b^{\ddagger} + kT \ln \dfrac{h}{kT}\left(\dfrac{E}{at} + \mathbf{k}_h\right)\right]\left(-\Delta G_h^{\ddagger} - kT \ln \dfrac{\mathbf{k}\,h}{kT}\right)\right\}^{1/2}} \\ \text{if} \quad \dfrac{atkT}{h}\left[\exp\left(-\dfrac{\Delta G_b^{\ddagger} - \mathbf{K}\,\alpha^2}{kT}\right) - \exp\left(-\dfrac{\Delta G_h^{\ddagger} + \mathbf{K}\,\alpha^2}{kT}\right)\right] < E \\ \qquad \leqq \dfrac{atkT}{h}\left[\exp\left(-\dfrac{\Delta G_b^{\ddagger} - \mathbf{K}\,\beta^2}{kT}\right) - \exp\left(-\dfrac{\Delta G_h^{\ddagger} + \mathbf{K}\,\beta^2}{kT}\right)\right], \\ 0 \quad \text{elsewhere}. \end{cases} \tag{26}$$

(b) Consider the uniformly distributed mechanical work over the domain $\{\alpha \div \beta\}$:

$$pdf_W(W) = \begin{cases} 1/(\beta - \alpha) & \text{if} \quad \alpha < W \leqq \beta \\ 0 & \text{elsewhere}, \end{cases} \tag{27}$$

and assume that $\mathbf{k}_h$ is negligibly small ($\mathbf{k}_h \ll \mathbf{k}_b$). Then, (10) becomes

$$pdf_{\mathbf{k}_b}(\mathbf{k}_b) = \begin{cases} \left|\dfrac{dF(\mathbf{k}_b)}{d\mathbf{k}_b}\right| pdf_W[F_b(\mathbf{k}_b)] = \dfrac{kT}{\mathbf{k}_b} \dfrac{1}{\beta - \alpha} \\ \quad \text{if} \quad \dfrac{kT}{h} \exp\left(-\dfrac{\Delta G_b^{\ddagger} - \alpha}{kT}\right) < \mathbf{k}_b \leqq \dfrac{kT}{h} \exp\left(-\dfrac{\Delta G_b^{\ddagger} - \beta}{kT}\right), \\ 0 \quad \text{elsewhere}, \end{cases} \tag{28}$$

and (18) reduces to

$$pdf_{E^*}(E^*) = \begin{cases} \dfrac{1}{t}\, pdf_{\mathbf{k}_b}\left(\dfrac{E^*}{t}\right) = \dfrac{kT}{E^*} \dfrac{1}{\beta - \alpha} \\ \text{if} \quad \dfrac{tkT}{h} \exp\left(-\dfrac{\Delta G_b^{\ddagger} - \alpha}{kT}\right) < E^* \leqq \dfrac{tkT}{h} \exp\left(-\dfrac{\Delta G_b^{\ddagger} - \beta}{kT}\right), \\ 0 \quad \text{elsewhere}, \end{cases}$$

where $E^* = E/a = t\,\mathbf{k}_b$.

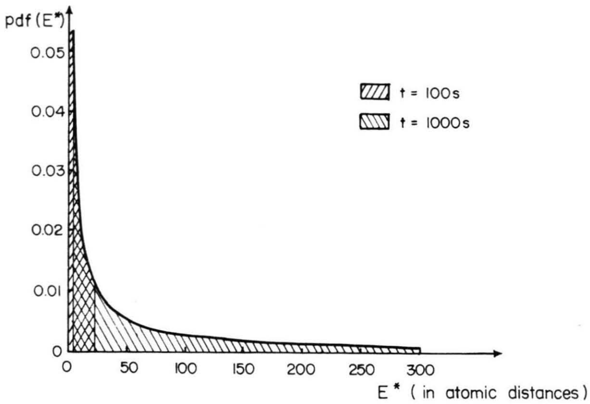

Fig. 1. Probability density function of the average crack size E^* at $t = 100$ s and $t = 1000$ s ($T = 300$ K, $\alpha = 0.8$, $\beta = 0.9$).

Equation (29) is a hyperbolic segment distribution of E^*. The limits of the segment vary with time, and, as the time increases, the tendency is toward non-zero probability for higher average crack sizes. The curve of Fig. 1 was calculated with (29), and the shaded areas show the probability distribution function for two different times.

Appendix A

The Probability Law of a Function of a Random Variable [7]

Consider a continuous random variable X with the probability distribution function $pdf_X(X)$. Assume that the differentiable monotonic function

$$Y = g(X) \tag{A-1}$$

has the inverse function

$$X = G(Y) \tag{A-2}$$

and the derivative

$$\frac{dX}{dY} = \frac{d}{dY} G(Y) = \left[\left| \frac{d}{dX} g(X) \right|_{X=G(Y)} \right]^{-1}. \tag{A-3}$$

In this case, (pdf_Y) is

$$pdf_Y(Y) = \left| \frac{d}{dY} G(Y) \right| pdf_X[G(Y)]. \tag{A-4}$$

Appendix B

The Distribution Law of a Sum of Independent Random Variables [8]

Assume that the s-independent random variables X and Y have the probability distribution functions $pdf_X(X)$ and $pdf_Y(Y)$, respectively.

The distribution of the sum

$$Z = X + Y \tag{B-1}$$

is given as

$$\begin{aligned} cdf_Z(Z) &= P\{X + Y \leq Z\} \\ &= \iint_{X+Y \leq Z} pdf_X(X)\, pdf_Y(Y)\, dX\, dY \\ &= \int_{-\infty}^{\infty} dX \int_{-\infty}^{Z-X} pdf_X(X)\, pdf_Y(Y)\, dY. \end{aligned} \tag{B-2}$$

By introducing the variable

$$Y = U - X, \tag{B-3}$$

(B-2) becomes

$$cdf_Z(Z) = \int_{-\infty}^{\infty} dX \int_{-\infty}^{Z} pdf_X(X)\, pdf_Y(U - X)\, dU, \tag{B-4}$$

and the differentiation of (B-4) results in the convolution integral

$$pdf_Z(Z) = \int_{-\infty}^{\infty} pdf_X(X)\, pdf_Y(Z - X)\, dX. \tag{B-5}$$

[1] P. C. Paris and F. Erdogan, J. Basic Engineering, Ser. D. **85** (4), 528 (1963).
[2] J. Carlsson, Probabilistic Fracture Mechanics, in "Advances in Elasto-Plastic Fracture Mechanics", edited by L. H. Larsson, Applied Science Publishers Ltd., London 1979.
[3] R. Tabreja, Engin. Fract. Mech. **11,** 839 (1979).
[4] S. Aoki and M. Sakata, Internat. J. Fract. **16,** No. 5 (1980).
[5] F. Kozin and J. L. Bogdanoff, Engin. Fract. Mech. **14,** 58 (1981).
[6] A. S. Krausz, K. Krausz, and D. Necsulescu, Z. Naturforsch. **38 a,** 497 (1983).
[7] E. Parzen, Modern Probability Theory and Its Applications, John Wiley, New York 1960.
[8] B. V. Gnedenko, The Theory of Probability, Chelsea Publishing Co., New York 1962.